4.7 – Change of Basis

Note: Recall that $(\mathbf{v})_B$ is the coordinate vector for a vector $\mathbf{v}$ relative to a basis B. When considered as a column vector, this author uses the notation $[\mathbf{v}]_B$.

Definition: If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a finite-dimensional vector space V, and if $(\mathbf{v})_S = (c_1, c_2, \ldots, c_n)$ is the coordinate vector of $\mathbf{v}$ relative to S, then the mapping $\mathbf{v} \to (\mathbf{v})_S$ is the **coordinate map relative to** S from V to R^n.

The standard basis for M_{22} is

$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. The coordinate

vector for $A = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$ ← in M_{22} relative to S

is $(A)_S = (6, 2, 5, 3)$ ← in R^4.

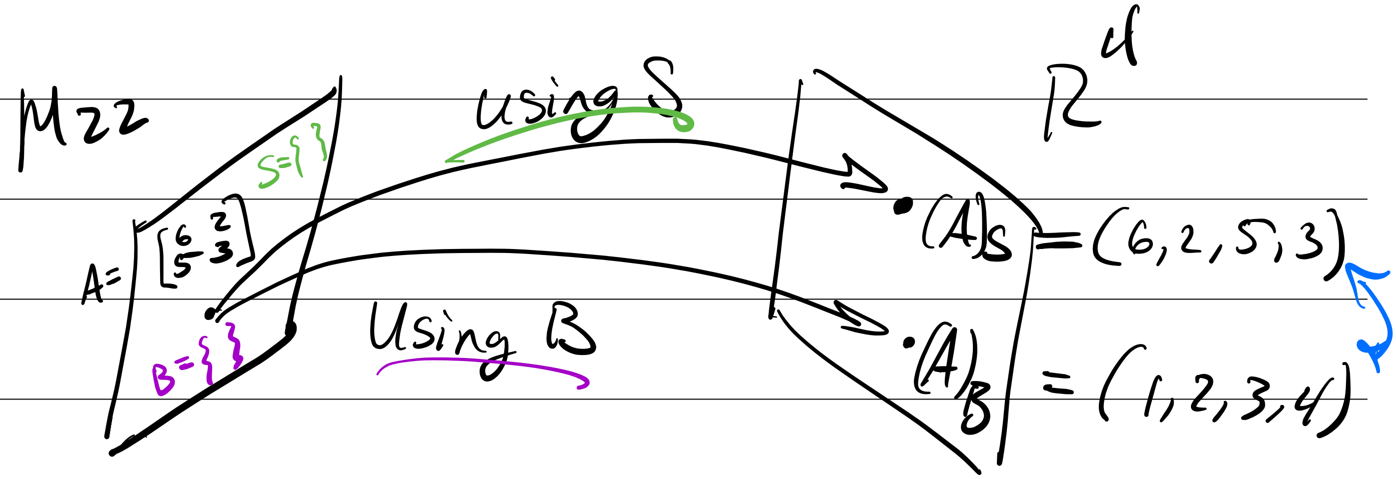
The coordinate vector relative to

$B = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ is

$(A)_B = (1, 2, 3, 4)$, a vector in R^4.

The coefficients change because the basis is different

This produces a mapping from M_{22} to R^4:



The goal of this section is to provide a link between $(A)_S$ and $(A)_B$, or more generally, between $(\vec{v})_B$ and $(\vec{v})_{B'}$.

old basis — new basis

To transition from one basis B to another basis B' for the same vector space, we use a transition matrix, denoted by $P_{B\to B'}$.

In 4.5, we found $(\mathbf{v})_{B'}$ for $\mathbf{v} = (-1,\ 7,\ 2)$ with respect to the basis $B' = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$, where $\mathbf{v}_1 = (2,\ -1,\ -1)$, $\mathbf{v}_2 = (-2,\ 1,\ 2)$, and $\mathbf{v}_3 = (3,\ 5,\ 4)$ by row reducing an augmented matrix. Here we achieve the same result by using a transition matrix from ~~S~~ B, the standard basis for R^3, to B'.

To find a relationship between B and B', we start by finding a linear combination of vectors in B' that map to vectors in B:

$$a\begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} + b\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} + c\begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{OR} \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This is 3 systems we can solve simultaneously:

$$\left[\begin{array}{ccc|ccc} 2 & -2 & 3 & 1 & 0 & 0 \\ -1 & 1 & 5 & 0 & 1 & 0 \\ -1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6/13 & -14/13 & 1 \\ 0 & 1 & 0 & 1/13 & -11/13 & 1 \\ 0 & 0 & 1 & 1/13 & 2/13 & 0 \end{array}\right]$$

Columns are vectors in B' (new basis)

Columns are vectors in B (old basis)

Maps B' to $\vec{e}_1$ ..... to $\vec{e}_2$ ... to $\vec{e}_3$

The resulting matrix on the right maps a vector relative to B to another vector relative to B'.

Consider $\vec{v} = \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix}$. This is $[\vec{v}]_B$.

$$\begin{bmatrix} 6/13 & -14/13 & 1 \\ 1/13 & -11/13 & 1 \\ 1/13 & 2/13 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ -4 \\ 1 \end{bmatrix} = [\vec{v}]_{B'}$$

Thus, $\begin{bmatrix} -6 \\ -4 \\ 1 \end{bmatrix} = [\vec{v}]_{B'} = \begin{bmatrix} 6/13 & -14/13 & 1 \\ 1/13 & -11/13 & 1 \\ 1/13 & 2/13 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix}$

$$\Rightarrow [\vec{v}]_{B'} = P_{B \to B'} [\vec{v}]_B$$

Definition: If we change the basis for a vector space V from an old basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ to a new basis $B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_n\}$, then $[\mathbf{u}_i]_{B'}$ are the coordinate vectors for the old basis vectors relative to the new basis. The **transition matrix from B to B′** is the matrix having these vectors as its columns and is written [by this author] as $P_{B \to B'} = [[\mathbf{u}_1]_{B'} | [\mathbf{u}_2]_{B'} | \ldots | [\mathbf{u}_n]_{B'}]$. Note that this is a partitioned matrix.

We use this as follows: For every vector $\mathbf{v}$ in V,

$$[\mathbf{v}]_{B'} = P_{B \to B'} [\mathbf{v}]_B$$

#**3** Consider the bases $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $B' = \left\{\mathbf{u}'_1, \mathbf{u}'_2, \mathbf{u}'_3\right\}$ for R^3, where

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ and } \mathbf{u}'_1 = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}, \mathbf{u}'_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \mathbf{u}'_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

a. Find the transition matrix B to B'.

b. Compute the coordinate vector $[\mathbf{w}]_B$, where $\mathbf{w} = \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix}$ and use the transition matrix in part (a) to compute $[\mathbf{w}]_{B'}$.

c. Check your work by computing $[\mathbf{w}]_{B'}$ directly.

$[\vec{u}_1]_{B'}$ $[\vec{u}_2]_{B'}$ $[\vec{u}_3]_{B'}$

a) $$\left[\begin{array}{ccc|ccc} 3 & 1 & -1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & -1 & 2 \\ -5 & -3 & 2 & 1 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 2 & 5/2 \\ 0 & 1 & 0 & -2 & -3 & -1/2 \\ 0 & 0 & 1 & 5 & 1 & 6 \end{array}\right]$$

New basis vectors

old basis vectors

b) $$\left[\begin{array}{ccc|c} 2 & 2 & 1 & -5 \\ 1 & -1 & 2 & 8 \\ 1 & 1 & 1 & -5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & -9 \\ 0 & 0 & 1 & -5 \end{array}\right] \quad [\vec{w}]_B = \begin{bmatrix} 9 \\ -9 \\ -5 \end{bmatrix}$$

$$P_{B \to B'}[\vec{w}]_B = \begin{bmatrix} 3 & 2 & 5/2 \\ -2 & -3 & -1/2 \\ 5 & 1 & 6 \end{bmatrix} \begin{bmatrix} 9 \\ -9 \\ -5 \end{bmatrix} = \begin{bmatrix} -7/2 \\ 23/2 \\ 6 \end{bmatrix} = [\vec{w}]_{B'}$$

c) $\left[\begin{array}{ccc|c} 3 & 1 & -1 & -5 \\ 1 & 1 & 0 & 8 \\ -5 & -3 & 2 & -5 \end{array}\right] \rightarrow$ yields $\begin{bmatrix} -7/2 \\ 23/2 \\ 6 \end{bmatrix}$

Another way to verify: $9\vec{u}_1 - 9\vec{u}_2 - 5\vec{u}_3 \overset{\text{should}}{=} -\frac{7}{2}\vec{u}_1' + \frac{23}{2}\vec{u}_2' + 6\vec{u}_3'$

This process is generalized in the 2×2 case:

Let $B = \{\vec{u}_1, \vec{u}_2\}$ be the old basis and $B' = \{\vec{u}_1', \vec{u}_2'\}$ be the new basis.

First, find a coordinate vector that maps B' to B. That is, when applied to vectors in B', it returns vectors in B:

$$\vec{u}_1 = a\vec{u}_1' + b\vec{u}_2'$$
$$\vec{u}_2 = c\vec{u}_1' + d\vec{u}_2'$$

Now let $\vec{v}$ be a fixed vector in V (for which B, B' are bases).

In terms of B, $\vec{v} = k_1(a\vec{u}_1' + b\vec{u}_2') + k_2(c\vec{u}_1' + d\vec{u}_2')$

$$\vec{v} = (k_1 a + k_2 c)\vec{u}_1' + (k_1 b + k_2 d)\vec{u}_2'$$

The new coordinate vector for $\vec{v}$ relative to B' is $[\vec{v}]_{B'} = \begin{bmatrix} k_1 a + k_2 c \\ k_1 b + k_2 d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{matrix} k_1 \\ k_2 \end{matrix}$

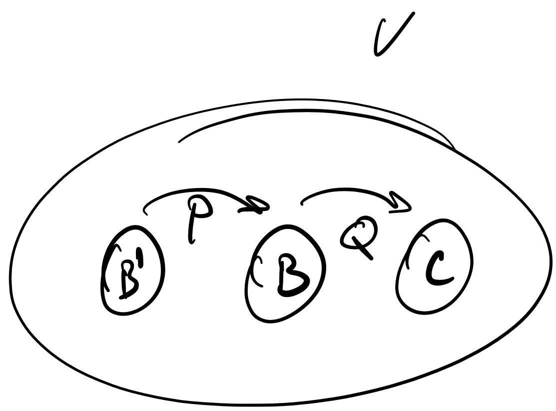
this is $[\vec{u}_1]_{B'}$

this is $[\vec{u}_2]_{B'}$

This matrix transitions from B to B'

Theorem 4.7.1 If P is the transition matrix from a basis B to a basis B' for a finite-dimensional vector space V, then P is invertible, and P^{-1} is the transition matrix from B' to B.

#13 If P is the transition matrix from a basis B' to a basis B, and Q is the transition matrix from B to a basis C, what is the transition matrix from B' to C? What is the transition matrix from C to B'?



PQ

$QP[\vec{v}]_{B'}$

$P_{B'\to C} = QP$

$P_{C\to B'} = P^{-1}Q^{-1}$